

Available online at www.sciencedirect.com

J. Math. Anal. Appl. 338 (2008) 317–329

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

On the exponential decay in thermoelasticity without energy dissipation and of type III in presence of an absorbing boundary [☆]

Barbara Lazzari, Roberta Nibbi *

Department of Mathematics, University of Bologna, 5 Piazza di Porta S. Donato, 40126 Bologna, Italy

Received 22 March 2007

Available online 18 May 2007

Submitted by Goong Chen

Abstract

We study the asymptotic behavior of the solution of a 3D hyperbolic system arising in the Green–Naghdi models of thermoelasticity of type II and III with a dissipative boundary condition for the displacement and prove that the energy exponentially decays in time.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Thermoelasticity without energy dissipation; Thermoelasticity of type III; Boundary conditions of memory type; Exponential asymptotic stability; Integro-differential equations

1. Introduction

The classical model of thermoelasticity, constructed on the basis of the Fourier law, provides good approximations for the description in a wide range of engineering applications. However it leads to the paradox of the infinite propagation speed of heat pulse and in some practical situations may lead to an inadequate description of heat conduction (see, for example, [4]). In order to eliminate these shortcomings of classical thermoelasticity many hyperbolic thermoelastic models have been developed from the middle of the last century. The literature dedicated to hyperbolic thermoelasticity is quite large and extensive reviews of works in this area are given in [5,16,17].

Here we consider thermoelastic models based on the theory developed by Green and Naghdi [13–15]. Instead of the classical entropy inequality, they use a general entropy balance and, introducing a new thermal variable, propose three models, based on the different material responses, labeled as type I, II and III. The linearized version of the first model leads to the Fourier law, the linearized version of both types II and III models allows heat transmission at finite speed. For the thermoelasticity of type II, or without energy dissipation, several results have been obtained

[☆] Research performed under the auspices of G.N.F.M. – I.N.d.A.M. and partially supported by Italian M.I.U.R. within the project “Modelli e metodi matematici nella fisica del continuo” and by the University of Bologna within the project “Modelli matematici di transizione di fase per sistemi complessi.”

* Corresponding author. Fax: +390512094490.

E-mail addresses: lazzari@dm.unibo.it (B. Lazzari), nibbi@dm.unibo.it (R. Nibbi).

about existence, uniqueness, continuous dependence, spatial decay and wave propagation (see [6,7,14,18,23,27,28]). For the thermoelasticity of type III, which presents thermal dissipation, using the classical energy method and the spectral method interesting results on the exponential stability in one space dimension [32] and in two or three space dimensions under suitable hypotheses on the domain or for symmetric solutions have been obtained in [29] and [35]. As for the classical thermoelasticity, also in this case the thermal dissipation does not assure the uniform decay of the energy for the general problem in three space variables [35].

Moreover, for both these models, the uniqueness and exponential growth of the solutions have been examined [28, 30] when the elasticity or the conductivity tensors are not positive definite and energy bounds for a class of non-standard problems are derived in [31]. Finally, very recently, the well posedness, L^p – L^q decays estimates and propagation of singularities of solution to the Cauchy problem in \mathbf{R}^3 have been considered in [34].

In this paper we study the long time behavior of the 3D linear thermoelastic systems of type II and III in presence of the following dissipative boundary condition with memory:

$$\mathbf{T}(x, t)n(x) = -\gamma_0 v(x, t) - \int_0^\infty \lambda(s) v^t(x, s) ds, \quad (1.1)$$

where \mathbf{T} is the stress tensor, n the unit outward normal vector, v the velocity and $v^t(x, s) := v(x, t - s)$ is the history of v .

Models of boundary conditions including a memory term which produces damping were proposed in [1] for the study of one-dimensional wave propagation, in [26] for sound evolution in a compressible fluid and in [10] in the context of Maxwell equations.

Recently in elasticity (see, for instance, [2,3,21] and references therein) and in electromagnetism [24] several authors have studied the dynamic problem associated to boundary conditions with memory terms of type (1.1) and have obtained results of existence, uniqueness and regularity of solutions, pointing out conditions on the memory kernel sufficient for the exponential or polynomial decay of the energy.

The boundary condition (1.1) allows the exponential decay of the energy also for the classical linear thermoelastic system and for the Cattaneo–Maxwell and Gurtin–Pipkin thermoelastic models [11,20] whenever the memory kernel decays exponentially. Here we obtain the same result for the Green–Naghdi thermoelastic models of type II and III.

It is interesting to observe that for the Green–Naghdi model of type II the mechanical boundary dissipation guarantees the exponential decay of the total energy (mechanical and thermal) even if there is no internal dissipation and no boundary damping for the temperature.

The outline of the paper is the following.

Section 2 is devoted to the Green–Naghdi model of type II. More precisely, in Section 2.1 the problem is set up. In Section 2.2 we prove existence, uniqueness and regularity of solutions for the relative initial boundary problem. In Section 2.3, after developing the needed estimates, we prove the exponential decay of the energy.

Finally, in Section 3, we briefly recall the linear thermoelastic system of type III and generalize the results of the previous section to this model.

2. The Green–Naghdi model of type II

2.1. Setup of the problem

Let Ω be a bounded open set of \mathbf{R}^3 with regular boundary $\partial\Omega$.

We put ourselves in the context of linear thermoelasticity, therefore the strain tensor will be approximated by the infinitesimal strain tensor $\mathbf{E} = \frac{1}{2}(\nabla u + \nabla u^T)$, where u is the displacement vector and θ will denote the temperature relative to Θ_0 , i.e., $\theta = \Theta - \Theta_0$, where Θ represents the absolute temperature. The stress tensor follows the constitutive equation

$$\mathbf{T} = \mathbf{C}\mathbf{E} - \alpha\mathbf{I}\theta,$$

the heat flux follows the law proposed by Green–Naghdi

$$\dot{q} = -\kappa^* \nabla \theta$$

and the rate at which the heat is absorbed follows the constitutive equation

$$h = c\theta + \beta \nabla \cdot u.$$

Here \mathbf{C} is a constant, fourth-order, symmetric tensor which is positive defined, i.e., there exist two positive constants k_1 and k_2 such that, for all symmetric second-order tensors \mathbf{B} ,

$$k_1 |\mathbf{B}|^2 \leq \mathbf{CB} \cdot \mathbf{B} \leq k_2 |\mathbf{B}|^2, \quad (2.1)$$

\mathbf{I} is the identity tensor, while c, κ^* are positive constants as well as $\frac{\alpha}{\beta}$.

The dynamic problem is described by the equation of motion

$$\rho \dot{v}(x, t) = \nabla \cdot [\mathbf{C} \nabla u(x, t) - \alpha \mathbf{I} \theta(x, t)] + f(x, t), \quad (2.2)$$

where $\rho > 0$ is the (constant) mass density and f is the force, the energy equation

$$c \dot{\theta}(x, t) = -\nabla \cdot [q(x, t) + \beta v(x, t)] + r(x, t), \quad (2.3)$$

where r is the external source and the Green–Naghdi law

$$q(x, t) = -\kappa^* \nabla \tau(x, t), \quad (2.4)$$

where τ is a new variable, called thermal displacement, which satisfies $\dot{\tau} = \theta$.

As for the boundary condition we assume that (1.1) is satisfied on $\partial\Omega$ together with a Neumann condition for the heat flux, that is

$$\mathbf{T}(x, t) n(x) = -\gamma_0 v(x, t) - \int_0^\infty \lambda'(s) w^t(x, s) ds, \quad q(x, t) \cdot n(x) = 0, \quad x \in \partial\Omega, \quad (2.5)$$

where condition (2.5)₁ is obtained from (1.1) by an integration by parts and $w^t(x, s) := u^t(x, s) - u(x, t)$ denotes the *past history* of u and is defined for $s \in \mathbf{R}^+$.

Moreover the memory kernel $\lambda: \mathbf{R}^+ \rightarrow \mathbf{R}$ belongs to $L^1(\mathbf{R}^+) \cap H^2(\mathbf{R}^+)$ and, by assuming that $\partial\Omega$ is locally strongly dissipative [2], we have

$$\gamma_0 \in \mathbf{R}^+, \quad \omega \int_0^\infty \lambda'(s) \sin(\omega s) ds < 0, \quad \forall \omega \neq 0.$$

Finally, we set the initial conditions

$$\begin{cases} u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \\ \tau(x, 0) = \tau_0(x), \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (2.6)$$

From now on we shall drop the x variable whenever no ambiguity arises. In the sequel, we shall refer to problem (2.2)–(2.6) as to *problem P*.

2.2. Well posedness

In this subsection we shall prove the well posedness of *problem P* via semigroup theory. Usually the solutions are found among those with “finite energy.” Here, due to the presence of a boundary condition with memory, the space where the solution of *problem P* is to be found depends on the family of histories with finite boundary free energy [2]. As it is well known it is possible to define several free energies. If we assume that the memory kernel satisfies the more restrictive hypotheses

$$\lambda'(s) < 0, \quad \lambda''(s) \geq 0, \quad \forall s \in \mathbf{R}^+, \quad (2.7)$$

it is possible to define the Graffi-type free energy [12] and the one introduced in [9]. Here we will work in the context of the latter free energy since, as it has been shown in [2], the corresponding space of admissible histories is larger with respect to the Graffi one.

Following [9], we define for any $s \in \mathbb{R}^+$,

$$\check{a}^t(s) := - \int_0^\infty \lambda'(\tau + s) w^t(\tau) d\tau,$$

so that the boundary condition (2.5)₁ takes the form

$$\check{\mathbf{T}}(t)n = \mathbf{T}(t)n + \gamma_0 v(t) = \check{a}^t(0) \quad (2.8)$$

and introduce the functional

$$\psi_{\partial\Omega}(t) := -\frac{1}{2} \int_{\partial\Omega} \int_0^\infty \frac{1}{\lambda'(s)} \frac{\partial \check{a}^t(s)}{\partial s} \cdot \frac{\partial \check{a}^t(s)}{\partial s} ds da$$

which satisfies

$$\begin{aligned} \dot{\psi}_{\partial\Omega}(t) = & - \int_{\partial\Omega} \check{\mathbf{T}}(t)n \cdot v(t) da \\ & - \frac{1}{2} \int_{\partial\Omega} \int_0^\infty \frac{\lambda''(s)}{[\lambda'(s)]^2} \frac{\partial \check{a}^t(s)}{\partial s} \cdot \frac{\partial \check{a}^t(s)}{\partial s} ds da + \frac{1}{2} \int_{\partial\Omega} \frac{1}{\lambda'(0)} \frac{\partial \check{a}^t(0)}{\partial s} \cdot \frac{\partial \check{a}^t(0)}{\partial s} da \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} \int_{\partial\Omega} |\check{a}^t(s)|^2 da &= \int_{\partial\Omega} \left| - \int_0^\infty \frac{\partial \check{a}^t(s+\tau)}{\partial \tau} d\tau \right|^2 da \\ &\leq \int_{\partial\Omega} \int_0^\infty -\lambda'(s+\tau) d\tau \int_0^\infty -\frac{1}{\lambda'(s+\tau)} \frac{\partial \check{a}^t(s+\tau)}{\partial \tau} \cdot \frac{\partial \check{a}^t(s+\tau)}{\partial \tau} d\tau da \leq 2\lambda(s) \psi_{\partial\Omega}(t). \end{aligned} \quad (2.10)$$

In particular, if we consider (2.8) and (2.10) for $s = 0$, we get

$$\int_{\partial\Omega} |\mathbf{T}(t)n|^2 da \leq 2\gamma_0^2 \int_{\partial\Omega} |v(t)|^2 da + 4\lambda(0) \psi_{\partial\Omega}(t). \quad (2.11)$$

Let us consider the state $\sigma = (v, \nabla u, \theta, \nabla \tau, \check{a}^t)$ and rewrite the *problem P* as an abstract first-order Cauchy problem as follows:

$$\begin{cases} \dot{\sigma}(t) = \mathbf{A}\sigma(t) + F(t), \\ \sigma(0) = \sigma_0 \end{cases} \quad (2.12)$$

with $F = (\frac{1}{\rho} f, 0, \frac{1}{c} r, 0, 0)$, $\sigma_0 = (v_0, \nabla u_0, \theta_0, \nabla \tau_0, \check{a}^o)$ and

$$\mathbf{A}\sigma = \left(\frac{1}{\rho} \nabla \cdot (\mathbf{C} \nabla u - \alpha \mathbf{I} \theta), \nabla v, -\frac{1}{c} \nabla \cdot (\beta v - \kappa^* \nabla \tau), \nabla \theta, \frac{\partial \check{a}^t(s)}{\partial s} - \lambda(s)v \right). \quad (2.13)$$

As said before, the natural setting in which to look for existence and uniqueness of solutions for problem (2.12) is the *admissible states space* \mathcal{K} , consisting in those states σ for which the total energy

$$\psi = \psi_\Omega + \psi_{\partial\Omega} = \frac{1}{2} \int_\Omega \left[\rho |v|^2 + \mathbf{C} \nabla u \cdot \nabla u + \frac{c\alpha}{\beta} \theta^2 + \frac{\kappa^* \alpha}{\beta} |\nabla \tau|^2 \right] dx + \psi_{\partial\Omega} \quad (2.14)$$

is finite. We endow \mathcal{K} with the inner product

$$\langle \sigma_1, \sigma_2 \rangle = \int_{\Omega} \left(\rho v_1 \cdot v_2 + \mathbf{C} \nabla u_1 \cdot \nabla u_2 + \frac{c\alpha}{\beta} \theta_1 \theta_2 + \frac{\kappa^* \alpha}{\beta} \nabla \tau_1 \cdot \nabla \tau_2 \right) dx - \int_{\partial\Omega} \int_0^{\infty} \frac{1}{\lambda'(s)} \frac{\partial \check{a}_1^t(s)}{\partial s} \cdot \frac{\partial \check{a}_2^t(s)}{\partial s} ds da,$$

where $\sigma_i = (v_i, \nabla u_i, \theta_i, \nabla \tau_i, \check{a}_i^t)$, for $i = 1, 2$, so that

$$\langle \sigma, \sigma \rangle = \|\sigma\|^2 = 2\psi(\sigma).$$

We denote by $\mathcal{D}(\mathbf{A})$ the domain of operator \mathbf{A} , namely

$$\mathcal{D}(\mathbf{A}) := \{\sigma \in \mathcal{K}; \mathbf{A}\sigma \in \mathcal{K} \text{ and boundary conditions (2.5) hold}\}$$

and claim that the operator \mathbf{A} is dissipative. In fact if $\sigma \in \mathcal{D}(\mathbf{A})$, we have

$$\begin{aligned} \langle \mathbf{A}\sigma, \sigma \rangle &= \int_{\Omega} [\nabla \cdot (\mathbf{C} \nabla u(t) - \alpha \mathbf{I} \theta(t)) \cdot v(t) + \mathbf{C} \nabla v(t) \cdot \nabla u(t)] dx \\ &\quad + \frac{\alpha}{\beta} \int_{\Omega} [\nabla \cdot (\kappa^* \nabla \tau(t) - \beta v(t)) \theta(t) + \kappa^* \nabla \theta(t) \cdot \nabla \tau(t)] dx \\ &\quad - \int_{\partial\Omega} \int_0^{\infty} \frac{1}{\lambda'(s)} \frac{\partial}{\partial s} \left[\frac{\partial \check{a}^t(s)}{\partial s} - \lambda(s) v(t) \right] \cdot \frac{\partial \check{a}^t(s)}{\partial s} ds da \\ &= \int_{\partial\Omega} \mathbf{T}(t) n \cdot v(t) da - \int_{\partial\Omega} \int_0^{\infty} \frac{1}{\lambda'(s)} \frac{\partial^2 \check{a}^t(s)}{\partial s^2} \cdot \frac{\partial \check{a}^t(s)}{\partial s} ds da - \int_{\partial\Omega} v(t) \cdot \check{a}^t(0) da \\ &= - \int_{\partial\Omega} \gamma_0 |v(t)|^2 da - \frac{1}{2} \int_{\partial\Omega} \int_0^{\infty} \frac{\lambda''(s)}{[\lambda'(s)]^2} \frac{\partial \check{a}^t(s)}{\partial s} \cdot \frac{\partial \check{a}^t(s)}{\partial s} ds da + \frac{1}{2} \int_{\partial\Omega} \frac{1}{\lambda'(0)} \frac{\partial \check{a}^t(0)}{\partial s} \cdot \frac{\partial \check{a}^t(0)}{\partial s} da \leq 0. \end{aligned}$$

We now proceed to show that also $\tilde{\mathbf{A}}$, the adjoint of \mathbf{A} , is dissipative so that, thanks to the Lumer–Phillips theorem, \mathbf{A} generates a C_0 -semigroup.

Let $\tilde{\sigma} = (\tilde{v}, \nabla \tilde{u}, \tilde{\theta}, \nabla \tilde{\tau}, \tilde{a}^t)$ be in \mathcal{K} and consider the boundary conditions

$$(\mathbf{C} \nabla \tilde{u}(t) - \alpha \mathbf{I} \tilde{\theta}(t)) n = \gamma_0 \tilde{u}(t) + \tilde{a}^t(0), \quad \nabla \tilde{\tau}(t) \cdot n = 0 \quad \text{on } \partial\Omega. \quad (2.15)$$

Denoting by H the Heaviside function and introducing a function $j(\tilde{a}^t)$ such that

$$\frac{\partial}{\partial s} j(\tilde{a}^t)(s) = -\lambda'(s) \frac{\partial}{\partial s} \left(\frac{H(s)}{\lambda'(s)} \right) \frac{\partial \tilde{a}^t(s)}{\partial s},$$

we claim that $\tilde{\mathbf{A}}\tilde{\sigma}$ is equal to

$$\left(\frac{1}{\rho} \nabla \cdot (\alpha \mathbf{I} \tilde{\theta} - \mathbf{C} \nabla \tilde{u}), -\nabla \tilde{v}, \frac{1}{c} \nabla \cdot (\beta \tilde{v} - \kappa^* \nabla \tilde{\tau}), -\nabla \tilde{\theta}, -\frac{\partial \tilde{a}^t(s)}{\partial s} + \lambda(s) \tilde{v} + j(\tilde{a}^t)(s) \right)$$

and that the domain of $\tilde{\mathbf{A}}$ is

$$\mathcal{D}(\tilde{\mathbf{A}}) := \{\tilde{\sigma} \in \mathcal{K}; \tilde{\mathbf{A}}\tilde{\sigma} \in \mathcal{K} \text{ and the boundary conditions (2.15) hold}\}.$$

Let us now compute $\langle \mathbf{A}\sigma, \tilde{\sigma} \rangle$, where $\sigma \in \mathcal{D}(\mathbf{A})$ and $\tilde{\sigma} \in \mathcal{D}(\tilde{\mathbf{A}})$:

$$\begin{aligned} \langle \mathbf{A}\sigma, \tilde{\sigma} \rangle &= \int_{\Omega} [\nabla \cdot (\mathbf{C} \nabla u(t) - \alpha \mathbf{I} \theta(t)) \cdot \tilde{v}(t) + \mathbf{C} \nabla v(t) \cdot \nabla \tilde{u}(t)] dx \\ &\quad - \frac{\alpha}{\beta} \int_{\Omega} [\nabla \cdot (\beta v(t) - \kappa^* \nabla \tau(t)) \tilde{\theta}(t) + \kappa^* \nabla \theta(t) \cdot \nabla \tilde{\tau}(t)] dx \end{aligned}$$

$$\begin{aligned}
& - \int_{\partial\Omega} \int_0^\infty \frac{1}{\lambda'(s)} \frac{\partial}{\partial s} \left(\frac{\partial \tilde{a}^t(s)}{\partial s} - \lambda(s)v(t) \right) \cdot \frac{\partial \tilde{a}^t(s)}{\partial s} \, ds \, da \\
& = - \int_{\Omega} [v(t) \cdot \nabla \cdot (\mathbf{C} \nabla \tilde{u}(t) - \alpha \mathbf{I} \tilde{\theta}(t)) + \mathbf{C} \nabla u(t) \cdot \nabla \tilde{v}(t)] \, dx \\
& \quad + \frac{\alpha}{\beta} \int_{\Omega} [\nabla \cdot (\beta \tilde{v}(t) - \kappa^* \nabla \tilde{\tau}(t)) \theta(t) + \kappa^* \nabla \tilde{\theta}(t) \cdot \nabla \tau(t)] \, dx - \frac{\kappa^* \alpha}{\beta} \int_{\partial\Omega} \theta(t) \nabla \tilde{\tau}(t) \cdot n \, da \\
& \quad + \int_{\partial\Omega} v(t) \cdot [(\mathbf{C} \nabla \tilde{u}(t) - \alpha \mathbf{I} \tilde{\theta}(t))n - \gamma_0 \tilde{v}(t) - \tilde{a}^t(0)] \, da \\
& \quad + \int_{\partial\Omega} \int_0^\infty \frac{1}{\lambda'(s)} \frac{\partial \tilde{a}^t(s)}{\partial s} \cdot \frac{\partial}{\partial s} \left(\frac{\partial \tilde{a}^t(s)}{\partial s} - \lambda(s)\tilde{v}(t) \right) \, ds \, da \\
& \quad + \int_{\partial\Omega} \int_0^\infty \frac{\partial}{\partial s} \left(\frac{1}{\lambda'(s)} \right) \frac{\partial \tilde{a}^t(s)}{\partial s} \cdot \frac{\partial \tilde{a}^t(s)}{\partial s} \, ds \, da + \int_{\partial\Omega} \frac{1}{\lambda'(0)} \frac{\partial \tilde{a}^t(0)}{\partial s} \cdot \frac{\partial \tilde{a}^t(0)}{\partial s} \, da,
\end{aligned}$$

so that, if $\tilde{\sigma}$ satisfies the boundary conditions (2.15), we have

$$\langle \mathbf{A}\sigma, \tilde{\sigma} \rangle = \langle \sigma, -\mathbf{A}\tilde{\sigma} \rangle + \int_{\partial\Omega} \int_{-\infty}^\infty \frac{\partial}{\partial s} \left(\frac{H(s)}{\lambda'(s)} \right) \frac{\partial \tilde{a}^t(s)}{\partial s} \cdot \frac{\partial \tilde{a}^t(s)}{\partial s} \, ds \, da.$$

Now observe that, for $\tilde{\sigma} \in \mathcal{D}(\tilde{\mathbf{A}})$, we have

$$\langle \tilde{\mathbf{A}}\tilde{\sigma}, \tilde{\sigma} \rangle = - \int_{\partial\Omega} \gamma_0 |\tilde{v}(t)|^2 \, da - \frac{1}{2} \int_{\partial\Omega} \int_0^\infty \frac{\lambda''(s)}{(\lambda'(s))^2} \frac{\partial \tilde{a}^t(s)}{\partial s} \cdot \frac{\partial \tilde{a}^t(s)}{\partial s} \, ds \, da \leq 0.$$

Finally, making use of the results obtained in [8], it is possible to state the following theorem establishing the well posedness of our problem in $\Omega \times \mathbb{R}^+$:

Theorem 2.1. *If $F \in W_{\text{loc}}^{1,p}(\mathbb{R}^+; L^2(\Omega))$ and $\sigma_0 \in \mathcal{D}(\mathbf{A})$, then problem (2.12) admits one and only one strict solution $\sigma \in C^1(\mathbb{R}^+; \mathcal{K}) \cap C(\mathbb{R}^+; \mathcal{D}(\mathbf{A}))$.*

2.3. Exponential decay

In order to show that an exponential decay of the energy (2.14) occurs over time, it is necessary to impose conditions on the memory kernel and, in addition, the domain Ω has to satisfy suitable regularity conditions. More precisely, we shall assume that

- (H1) (2.7) holds and there exists $k_0 > 0$ such that $\lambda''(s) + k_0 \lambda'(s) \geq 0$, $\forall s \geq 0$;
 (H2) Ω be *strongly star shaped* with respect to a point, namely there exists x_0 in Ω and $p > 0$ such that, letting $l(x) := x - x_0$,

$$l(x) \cdot n(x) \geq p |l(x)| > 0, \quad \forall x \in \partial\Omega.$$

The main result of this section is the following

Theorem 2.2. *Let σ be a solution of (2.12) with null sources and initial data $\sigma_0 \in \mathcal{D}(\mathbf{A})$. If (H1) and (H2) hold, there exist two positive constants c_1 and c_2 such that*

$$\psi(t) \leq c_2 e^{-c_1 t} \psi(0).$$

Following [19], we define the functional

$$\mathcal{L}_{t_0}(t) := (t + t_0)\psi_{\Omega}(t) + 2 \int_{\Omega} \left(\rho v(t) \cdot [\nabla u(t)l(x) + u(t)] + \frac{\alpha}{\beta} [c\theta(t) + \beta \nabla \cdot u(t)] [\nabla \tau(t) \cdot l(x) + \tau(t)] \right) dx,$$

where $l(x) = (x - x_0)$, and prove the following crucial result.

Lemma 2.1. *Under the hypotheses of Theorem 2.2 the following inequalities hold:*

- for t_0 sufficiently large

$$\frac{d}{dt} \mathcal{L}_{t_0}(t) \leq -\psi_{\Omega}(t) - \frac{d}{dt} \left[2 \int_{\partial\Omega} \int_0^{\infty} \check{a}^t(s) \cdot u(t) ds da + (t + t_0)\psi_{\partial\Omega}(t) + (\gamma_0 + \Lambda_0) \int_{\partial\Omega} |u(t)|^2 da \right], \quad (2.16)$$

where $\Lambda_0 := \int_0^{\infty} \lambda(s) ds$;

- there exist two positive constants h_1 and h_2 , depending on the initial data and on the domain Ω , such that

$$\mathcal{L}_{t_0}(T) - \mathcal{L}_{t_0}(0) \geq (T + t_0 - h_1)\psi_{\Omega}(T) - \frac{1}{2} \int_0^T \psi_{\Omega}(t) dt - h_2 \quad (2.17)$$

for all $T > 0$.

Proof. In order to prove (2.16), we observe that if σ is a solution of (2.12), then it is easy to show that

$$\dot{\psi}_{\Omega}(t) = \int_{\partial\Omega} \mathbf{T}(t)n \cdot v(t) da;$$

therefore, applying the divergence theorem and recalling the symmetry of \mathbf{C} , we have

$$\frac{d}{dt} \mathcal{L}_{t_0}(t) = -\psi_{\Omega}(t) + (t + t_0) \int_{\partial\Omega} \mathbf{T}(t)n \cdot v(t) da + I_{\partial\Omega}(t), \quad (2.18)$$

where

$$\begin{aligned} I_{\partial\Omega}(t) = & \int_{\partial\Omega} \left[\rho |v(t)|^2 + \frac{c\alpha}{\beta} \theta^2(t) - \mathbf{C} \nabla u(t) \cdot \nabla u(t) - \frac{\kappa^* \alpha}{\beta} |\nabla \tau(t)|^2 \right] l \cdot n da \\ & + 2 \int_{\partial\Omega} \mathbf{T}(t)n \cdot (\nabla u(t)l + u(t)) da + 2\alpha \int_{\partial\Omega} \theta(t) \nabla \cdot u(t) l \cdot n da. \end{aligned} \quad (2.19)$$

We now proceed to estimate the various terms in (2.18) and (2.19).

Recalling (2.8), (2.9) and (H1), we get

$$(t + t_0) \int_{\partial\Omega} \mathbf{T}(t)n \cdot v(t) da \leq -\gamma_0(t + t_0) \int_{\partial\Omega} |v(t)|^2 da - \frac{d}{dt} [(t + t_0)\psi_{\partial\Omega}(t)] + [1 - k_0(t + t_0)]\psi_{\partial\Omega}(t). \quad (2.20)$$

By (H2), (2.1) and (2.11) we have

$$\begin{aligned} 2 \int_{\partial\Omega} \mathbf{T}(t)n \cdot \nabla u(t)l da & \leq \int_{\partial\Omega} |l| \left[\frac{3}{pk_1} |\mathbf{T}(t)n|^2 + \frac{pk_1}{3} |\nabla u(t)|^2 \right] da \\ & \leq \frac{3}{pk_1} \int_{\partial\Omega} |l| |\mathbf{T}(t)n|^2 da + \frac{1}{3} \int_{\partial\Omega} (l \cdot n) \mathbf{C} \nabla u(t) \cdot \nabla u(t) da \end{aligned} \quad (2.21)$$

and

$$\begin{aligned}
2\alpha \int_{\partial\Omega} \theta(t) \nabla \cdot u(t) l \cdot n \, da &\leq \int_{\partial\Omega} l \cdot n \left[\frac{3\alpha^2}{k_1} \theta^2(t) + \frac{k_1}{3} |\nabla u(t)|^2 \right] da \\
&\leq \frac{3\alpha^2}{k_1} \int_{\partial\Omega} \theta^2(t) l \cdot n \, da + \frac{1}{3} \int_{\partial\Omega} (l \cdot n) \mathbf{C} \nabla u(t) \cdot \nabla u(t) \, da.
\end{aligned} \tag{2.22}$$

Moreover, thanks to (2.1), (2.8), (2.10) and (2.13)₅, we get

$$\begin{aligned}
2 \int_{\partial\Omega} \mathbf{T}(t) n \cdot u(t) \, da &= -\gamma_0 \frac{d}{dt} \int_{\partial\Omega} |u(t)|^2 \, da - 2 \int_{\partial\Omega} \int_0^\infty \frac{\partial \check{a}^t(s)}{\partial s} \cdot u(t) \, ds \, da \\
&= -\frac{d}{dt} \left[(\gamma_0 + \Lambda_0) \int_{\partial\Omega} |u(t)|^2 \, da + 2 \int_{\partial\Omega} \int_0^\infty \check{a}^t(s) \cdot u(t) \, ds \, da \right] + 2 \int_{\partial\Omega} \int_0^\infty \check{a}^t(s) \cdot v(t) \, ds \, da \\
&\leq \int_{\partial\Omega} |v(t)|^2 \, da + 2\Lambda_0 \psi_{\partial\Omega}(t) \\
&\quad - \frac{d}{dt} \left[(\gamma_0 + \Lambda_0) \int_{\partial\Omega} |u(t)|^2 \, da + 2 \int_{\partial\Omega} \int_0^\infty \check{a}^t(s) \cdot u(t) \, ds \, da \right],
\end{aligned} \tag{2.23}$$

while, recalling the definition of the stress tensor and (2.1), we have

$$\int_{\partial\Omega} \left(\alpha^2 \frac{3}{k_1} + \frac{c}{\alpha\beta} \right) \theta^2(t) l \cdot n \, da \leq \int_{\partial\Omega} \left[12k_2 \left(\frac{3}{k_1} + \frac{c}{\alpha^3\beta} \right)^2 |\mathbf{T}(t)n|^2 + \frac{1}{3} \mathbf{C} \nabla u(t) \cdot \nabla u(t) \right] l \cdot n \, da. \tag{2.24}$$

Therefore, substituting (2.20)–(2.24) in (2.18) and taking into account (2.11), we obtain

$$\begin{aligned}
\frac{d}{dt} \mathcal{L}_{t_0}(t) &\leq -\psi_{\Omega}(t) + [\delta_1 - \gamma_0(t + t_0)] \int_{\partial\Omega} |v(t)|^2 \, da + [\delta_2 - k_0(t + t_0)] \psi_{\partial\Omega}(t) \\
&\quad - \frac{d}{dt} \left[2 \int_{\partial\Omega} \int_0^\infty \check{a}^t(s) \cdot u(t) \, ds \, da + (t + t_0) \psi_{\partial\Omega}(t) + (\gamma_0 + \Lambda_0) \int_{\partial\Omega} |u(t)|^2 \, da \right]
\end{aligned}$$

for suitable positive constants δ_1 and δ_2 . By choosing t_0 sufficiently large we get (2.16).

We now want to prove (2.17).

Recalling the definition of $\mathcal{L}_{t_0}(t)$ and (2.1), we observe that

$$\begin{aligned}
&\left| 2 \int_{\Omega} \left[\rho v(t) \cdot \nabla u(t) l + \frac{\alpha}{\beta} (c\theta(t) + \beta \nabla \cdot u(t)) \nabla \tau(t) \cdot l \right] dx \right| \\
&\leq \int_{\Omega} |l| \left[\rho |v(t)|^2 + \left(\frac{\rho}{k_1} + \frac{\alpha\beta}{k_1\kappa^*} \right) \mathbf{C} \nabla u(t) \cdot \nabla u(t) + \frac{c\alpha}{\beta} \theta^2(t) + \left(1 + \frac{c}{\kappa^*} \right) \frac{\alpha\kappa^*}{\beta} |\nabla \tau(t)|^2 \right] dx \\
&\leq 2 \operatorname{diam}(\Omega) \max \left\{ \frac{\rho\kappa^* + \alpha\beta}{k_1\kappa^*}, \frac{\kappa^* + c}{\kappa^*} \right\} \psi_{\Omega}(t)
\end{aligned}$$

while, denoting with k_P the constant of Poincaré's inequality (see [33]), the following inequality

$$\begin{aligned}
&\left| 2 \int_{\Omega} \left[\rho v(t) \cdot u(t) + \frac{\alpha}{\beta} (c\theta(t) + \beta \nabla \cdot u(t)) \tau(t) \right] dx \right| \\
&\leq \frac{\rho}{\varepsilon_1} \int_{\Omega} |v(t)|^2 \, dx + \frac{c\alpha}{\beta\varepsilon_2} \int_{\Omega} \theta^2(t) \, dx + \left(\frac{\varepsilon_1 \rho k_P}{k_1} + \frac{1}{k_1 \varepsilon_3} \right) \int_{\Omega} \mathbf{C} \nabla u \cdot \nabla u \, dx
\end{aligned}$$

$$+ \frac{c\alpha}{\beta} k_P \left(\varepsilon_2 + \frac{\alpha\beta}{c} \varepsilon_3 \right) \int_{\Omega} |\nabla \tau(t)|^2 dx + \varepsilon_1 k_P \rho \left| \int_{\Omega} u(t) dx \right|^2 + \frac{c\alpha}{\beta} k_P \left(\varepsilon_2 + \frac{\alpha\beta}{c} \varepsilon_3 \right) \left| \int_{\Omega} \tau(t) dx \right|^2, \quad (2.25)$$

holds for any $\varepsilon_i > 0$ ($i = 1, 2, 3$).

Since

$$\begin{aligned} & \varepsilon_1 k_P \rho \left| \int_{\Omega} u(t) dx \right|^2 + \frac{c\alpha}{\beta} k_P \left(\varepsilon_2 + \frac{\alpha\beta}{c} \varepsilon_3 \right) \left| \int_{\Omega} \tau(t) dx \right|^2 \\ & \leq 2 \left[\varepsilon_1 k_P \rho \left| \int_{\Omega} u_0 dx \right|^2 + \frac{c\alpha}{\beta} k_P \left(\varepsilon_2 + \frac{\alpha\beta}{c} \varepsilon_3 \right) \left| \int_{\Omega} \tau_0 dx \right|^2 \right] + 4k_P \max \left\{ \varepsilon_1, \varepsilon_2 + \frac{\alpha\beta}{c} \varepsilon_3 \right\} \int_0^t \psi_{\Omega}(\tau) d\tau, \end{aligned} \quad (2.26)$$

if we choose $\varepsilon_1, \varepsilon_2$ and ε_3 such that

$$\frac{1}{8k_P} = \varepsilon_1 = 2\varepsilon_2 = 2\frac{\alpha\beta}{c}\varepsilon_3$$

and put (2.26) in (2.25), we obtain the thesis letting

$$h_1 = 2 \operatorname{diam}(\Omega) \max \left\{ \frac{\rho\kappa^* + \alpha\beta}{k_1\kappa^*}, \frac{\kappa^* + c}{\kappa^*} \right\} + 2 \max \left\{ 16k_P, \frac{\rho}{8k_1} + \frac{16k_P\alpha\beta}{ck_1}, \frac{c}{8\kappa^*} \right\}$$

and

$$h_2 = \frac{\rho}{2} \left| \int_{\Omega} u_0 dx \right|^2 + \frac{1}{2} \frac{c\alpha}{\beta} \left| \int_{\Omega} \tau_0 dx \right|^2 + (t_0 + h_1) \psi_{\Omega}(0). \quad \square$$

We are now able to proceed to prove the main result of this section.

Proof of Theorem 2.2. Taking t_0 sufficiently large and integrating (2.16) from 0 to T we get

$$\begin{aligned} \mathcal{L}_{t_0}(T) - \mathcal{L}_{t_0}(0) & \leq - \int_0^T \psi_{\Omega}(t) dt - (T + t_0) \psi_{\partial\Omega}(T) \\ & \quad - \int_{\partial\Omega} \left[2 \int_0^{\infty} \check{a}^T(s) ds \cdot u(T) + (\Lambda_0 + \gamma_0) |u(T)|^2 \right] da + h_3, \end{aligned} \quad (2.27)$$

where

$$h_3 = t_0 \psi_{\partial\Omega}(0) + 2 \int_{\partial\Omega} \int_0^{\infty} \check{a}^0(s) ds \cdot u(0) da + (\Lambda_0 + \gamma_0) \int_{\partial\Omega} |u(0)|^2 da.$$

On the other hand, by (2.10), we have

$$\begin{aligned} & - \int_{\partial\Omega} \left[2 \int_0^{\infty} \check{a}^T(s) ds \cdot u(T) + (\Lambda_0 + \gamma_0) |u(T)|^2 \right] da \\ & = - \int_{\partial\Omega} \left[\sqrt{\Lambda_0 + \gamma_0} u(T) + \frac{1}{\sqrt{\Lambda_0 + \gamma_0}} \int_0^{\infty} \check{a}^T(s) ds \right]^2 da + \frac{1}{\Lambda_0 + \gamma_0} \int_{\partial\Omega} \left| \int_0^{\infty} \check{a}^T(s) ds \right|^2 da \\ & \leq 2 \frac{\Lambda_0}{\Lambda_0 + \gamma_0} \psi_{\partial\Omega}(T); \end{aligned}$$

therefore, letting $h_4 = 2\frac{A_0}{A_0 + \gamma_0}$, (2.27) becomes

$$\mathcal{L}_{t_0}(T) - \mathcal{L}_{t_0}(0) \leq - \int_0^T \psi_{\Omega}(t) dt - [T + t_0 - h_4] \psi_{\partial\Omega}(T) + h_3. \quad (2.28)$$

Comparing (2.17) and (2.28), letting $h = \max\{h_1, h_4\}$, we obtain

$$\psi(T) \leq \frac{h_2 + h_3}{T + t_0 - h}. \quad (2.29)$$

Estimate (2.29) ensures the exponential decay of ψ thanks to the semigroup properties proved in the preceding section (see, for instance, [25, Theorem 4.1]). \square

We close this subsection observing that the exponential decay of the boundary memory kernel turns out to be a necessary condition for the exponential decay of the boundary free energy $\psi_{\partial\Omega}$. To be more precise we give the following

Definition 2.1. We say that a function f *decays exponentially* if there exists a positive constant γ such that

$$\int_0^\infty e^{\gamma t} |f(t)| dt < \infty.$$

And state our result:

Proposition 2.1. Let σ be a solution of problem P with null sources and such that

$$0 < \int_0^\infty \int_{\partial\Omega} e^{2\gamma t} |v(t)|^2 da dt < \infty, \quad \int_0^\infty \int_{\partial\Omega} e^{2\gamma t} \psi_{\partial\Omega}(t) n da dt < \infty \quad (2.30)$$

for some $\gamma > 0$. If (2.7) and (H2) hold, then λ decays exponentially.

Remark 2.1. The hypothesis (2.30), thanks to (2.11), guarantees the exponential decay of the $L^2(\partial\Omega)$ -norm of $\mathbf{T}n$ and Proposition 2.1 is a plain consequence of a theorem by Murakami [22, Theorem 2] as shown in [24]. Moreover, Proposition 2.1 can be improved by asking directly the exponential decay of the $L^2(\partial\Omega)$ -norm of $\mathbf{T}n$ instead of (2.30)₂ (see, for instance, [2]).

3. The Green–Naghdi model of type III

Here we extend the results obtained in the previous section to the model of type III characterized by the following constitutive equation for the heat flux:

$$q = -\kappa^* \nabla \tau - \tilde{\kappa} \nabla \theta,$$

where κ^* and $\tilde{\kappa}$ are positive constants.

For easiness in writing we suppose that the memory kernel in (1.1) vanishes, so that we associate to the evolutive problem the following boundary conditions:

$$\mathbf{T}(x, t)n(x) = -\gamma_0 v(x, t), \quad \tau(x, t) = 0, \quad x \in \partial\Omega, \quad (3.1)$$

while the initial conditions are given by (2.6).

The evolutive problem \tilde{P} can be rewritten as a first-order Cauchy problem

$$\begin{cases} \dot{\sigma}(t) = \mathbf{A}\sigma(t) + F(t), \\ \sigma(0) = \sigma_0 \end{cases} \quad (3.2)$$

by choosing $\sigma = (v, \nabla u, \theta, \nabla \tau)$, $F = (\frac{1}{\rho}f, 0, \frac{1}{c}r, 0)$, $\sigma_0 = (v_0, \nabla u_0, \theta_0, \nabla \tau_0)$,

$$\mathbf{A}\sigma = \left(\frac{1}{\rho} \nabla \cdot (\mathbf{C} \nabla u - \alpha \mathbf{I} \theta), \nabla v, \frac{1}{c} \nabla \cdot (\kappa^* \nabla \tau + \tilde{\kappa} \nabla \theta - \beta v), \nabla \theta \right)$$

and

$$\mathcal{D}(\mathbf{A}) := \{ \sigma \in H^1(\Omega) \times H^*(\Omega) \times H^2(\Omega) \times H^*(\Omega); \text{ boundary conditions (3.1) hold} \},$$

where $H^*(\Omega) := \{ \Phi \in L^2(\Omega); \nabla \cdot \Phi \in L^2(\Omega) \}$.

By introducing the energy

$$\psi_\Omega = \frac{1}{2} \int_\Omega \left[\rho |v|^2 + \mathbf{C} \nabla u \cdot \nabla u + \frac{c\alpha}{\beta} \theta^2 + \frac{\kappa^* \alpha}{\beta} |\nabla \tau|^2 \right] dx,$$

the operator domain is a dense subset of the state space with finite energy and it is possible to show that *problem \tilde{P}* is well posed since the operator \mathbf{A} generates a C_0 -semigroup [35]. We now want to prove that if σ is a solution to *problem \tilde{P}* with null sources, then the energy exponentially decays in time. To this end, in a similar way as in the previous section, we introduce the functional

$$\mathcal{M}_{t_0}(t) := (t + t_0) \psi_\Omega(t) + 2 \int_\Omega \left(\rho v(t) \cdot [\nabla u(t) l(x) + u(t)] + \frac{\alpha}{\beta} [c \theta(t) + \beta \nabla \cdot u(t)] \tau(t) \right) dx,$$

where $l(x) = (x - x_0)$ and the exponential decay can be obtained as a direct consequence of the following

Lemma 3.1. *If the domain Ω is strongly star shaped with respect to x_0 and σ is a solution of (3.2) with null sources and initial data $\sigma_0 \in \mathcal{D}(\mathbf{A})$, the following inequalities hold:*

- for t_0 sufficiently large

$$\frac{d}{dt} \mathcal{M}_{t_0}(t) \leq -\frac{1}{2} \psi_\Omega(t) - \gamma_0 \frac{d}{dt} \left[\int_{\partial\Omega} |u(t)|^2 da \right]; \quad (3.3)$$

- there exist positive constants γ_1 and γ_2 , depending on the initial data and on the domain Ω , such that

$$\mathcal{M}_{t_0}(T) - \mathcal{M}_{t_0}(0) \geq (T + t_0 - \gamma_1) \psi_\Omega(T) - \frac{1}{2} \int_0^T \psi_\Omega(t) dt - \gamma_2 \quad (3.4)$$

for all $T > 0$.

Proof. In order to prove (3.3), we observe that, if σ is a solution of (3.2), then it is easy to show that

$$\dot{\psi}_\Omega(t) = \int_{\partial\Omega} \left(\mathbf{T}(t)n \cdot v(t) - \frac{\alpha}{\beta} \theta(t) q(t) \cdot n \right) da - \int_\Omega \tilde{\kappa} \nabla \theta(t) \cdot \nabla \theta(t) dx;$$

therefore, applying the divergence theorem and recalling the symmetry of \mathbf{C} , we have

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{t_0}(t) = & -\psi_\Omega(t) + (t + t_0) \left[\int_{\partial\Omega} \left(\mathbf{T}(t)n \cdot v(t) - \frac{\alpha}{\beta} \theta(t) q(t) \cdot n \right) da - \int_\Omega \tilde{\kappa} \nabla \theta(t) \cdot \nabla \theta(t) dx \right] \\ & - \frac{\alpha \kappa^*}{\beta} \int_\Omega \nabla \tau(t) \cdot \nabla \tau(t) dx + 3 \frac{c\alpha}{\beta} \int_\Omega \theta^2(t) dx + 4\alpha \int_\Omega \theta(t) \nabla \cdot u(t) dx + J_{\partial\Omega}(t), \end{aligned} \quad (3.5)$$

where

$$\begin{aligned}
J_{\partial\Omega}(t) &= \int_{\partial\Omega} [\rho|v(t)|^2 - \mathbf{C}\nabla u(t) \cdot \nabla u(t)] l \cdot n \, da + 2 \int_{\partial\Omega} \mathbf{C}\nabla u(t)n \cdot (\nabla u(t)l + u(t)) \, da \\
&\quad + 4\alpha \int_{\partial\Omega} \theta(t)u(t) \cdot n \, da - \frac{2\alpha}{\beta} \int_{\partial\Omega} \tau(t)q(t) \cdot n \, da.
\end{aligned} \tag{3.6}$$

Thanks to (2.1) and (3.1), we have

$$4\alpha \int_{\Omega} \theta(t) \nabla \cdot u(t) \, dx \leq 16\alpha^2 k_1 \int_{\Omega} \theta^2(t) \, dx + \frac{1}{4} \int_{\Omega} \mathbf{C}\nabla u(t) \cdot \nabla u(t) \, dx$$

and

$$\begin{aligned}
J_{\partial\Omega}(t) &= \int_{\partial\Omega} [\rho|v(t)|^2 - \mathbf{C}\nabla u(t) \cdot \nabla u(t)] l \cdot n \, da + 2 \int_{\partial\Omega} \mathbf{T}(t)n \cdot (\nabla u(t)l + u(t)) \, da \\
&\leq -\gamma_0 \frac{d}{dt} \left[\int_{\partial\Omega} |u(t)|^2 \, da \right] + \int_{\partial\Omega} \left(\rho + \frac{1}{pk_1} \right) |l| |v(t)|^2 \, da.
\end{aligned}$$

Therefore, applying the Poincaré inequality (with constant k_p) to (3.5), we get

$$\begin{aligned}
\frac{d}{dt} \mathcal{M}_{t_0}(t) &\leq -\frac{1}{2} \psi_{\Omega}(t) - \gamma_0 \frac{d}{dt} \left[\int_{\partial\Omega} |u(t)|^2 \, da \right] - \gamma_0(t + t_0 - \gamma_3) \int_{\partial\Omega} |v(t)|^2 \, da \\
&\quad - (t + t_0 - \gamma_4) \int_{\Omega} \tilde{\kappa} \nabla \theta(t) \cdot \nabla \theta(t) \, dx,
\end{aligned}$$

where

$$\gamma_3 = \frac{\text{diam}(\Omega)}{\gamma_0} \left(\rho + \frac{1}{pk_1} \right), \quad \gamma_4 = \frac{k_p}{\tilde{\kappa}} \left(16\alpha^2 k_1 + \frac{3c\alpha}{\beta} \right)$$

and (3.3) is proved by choosing $t_0 \geq \max\{\gamma_3, \gamma_4\}$.

We now want to prove (3.4). Reasoning in the same way as in the proof of (2.17), we obtain the thesis letting

$$\gamma_1 = 2 \, \text{diam}(\Omega) \max \left\{ 1, \frac{\rho}{k_1} \right\} + 2 \max \left\{ 8k_p, \frac{\rho}{8k_1} + \frac{1}{k_1}, 1, \alpha^2 \frac{k_p}{\kappa^*} \left(\frac{c}{\alpha\beta} + 1 \right) \right\}$$

and

$$\gamma_2 = \frac{\rho}{4} \left| \int_{\Omega} u_0 \, dx \right|^2 + (t_0 + \delta_1) \psi_{\Omega}(0). \quad \square$$

Remark 3.1. The exponential decay of the energy can also be obtained if the memory kernel does not vanish but, as observed in the previous section, satisfies (H1).

References

- [1] J. Bielak, R.C. MacCamy, Dissipative boundary conditions for one-dimensional wave propagation, *J. Integral Equations Appl.* 2 (1990) 307–331.
- [2] C.A. Bosello, B. Lazzari, R. Nibbi, A viscous boundary condition with memory in linear elasticity, *Internat. J. Engrg. Sci.* 45 (2007) 94–110.
- [3] M.M. Cavalcanti, A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition with memory type, *Differential Integral Equations* 18 (2005) 583–600.
- [4] G. Caviglia, A. Morro, B. Straughan, Thermoelasticity at cryogenic temperatures, *Internat. J. Non-Linear Mech.* 27 (1992) 251–263.
- [5] D.S. Chandrasekharaiah, Hyperbolic thermoelasticity: A review of recent literature, *Appl. Mech. Rev.* 51 (1998) 705–729.
- [6] D.S. Chandrasekharaiah, A note on the uniqueness of solution in the linear theory of thermoelasticity without energy dissipation, *J. Elasticity* 43 (1996) 279–283.

- [7] D.S. Chandrasekharaiah, One-dimensional wave propagation in the linear theory of thermoelasticity without energy dissipation, *J. Thermal Stresses* 19 (1996) 695–710.
- [8] G. Da Prato, E. Sinestrari, Differential operators with non dense domain, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 14 (1987) 285–344.
- [9] L. Deseri, M. Fabrizio, M. Golden, The concept of minimal state in viscoelasticity: new free energies and applications to PDEs, *Arch. Ration. Mech. Anal.* 181 (2006) 43–96.
- [10] M. Fabrizio, A. Morro, A boundary condition with memory in electromagnetism, *Arch. Ration. Mech. Anal.* 136 (1996) 359–381.
- [11] H. Gao, J.E. Munoz Rivera, On the exponential stability of thermoelastic problem with memory, *Appl. Anal.* 78 (2001) 379–403.
- [12] D. Gaffi, Sull'espressione dell'energia libera nei materiali viscoelastici lineari, *Ann. Mat. Pura Appl.* 98 (1974) 273–279.
- [13] A.E. Green, P.M. Naghdi, A re-examination of the basic postulates of themomechanics, *Proc. Roy. Soc. London A* 432 (1991) 171–194.
- [14] A.E. Green, P.M. Naghdi, On thermoelasticity without energy dissipation, *J. Elasticity* 31 (1993) 189–208.
- [15] A.E. Green, P.M. Naghdi, A unified procedure for construction of theories of deformable media, I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua, *Proc. Roy. Soc. London A* 448 (1995) 335–356, 357–377, 379–388.
- [16] R.B. Hetnarski, J. Ignaczak, Generalized thermoelasticity, *J. Thermal Stresses* 22 (1999) 451–470.
- [17] R.B. Hetnarski, J. Ignaczak, Nonclassical dynamical thermoelasticity, *Internat. J. Solids Structures* 37 (2000) 215–224.
- [18] D. Işan, On the theory of thermoelasticity without energy dissipation, *J. Thermal Stresses* 21 (1998) 295–307.
- [19] J.E. Lagnese, Decay of solutions of wave equations in a bounded region with boundary dissipation, *J. Differential Equations* 50 (1983) 163–182.
- [20] B. Lazzari, R. Nibbi, On the energy decay of a linear hyperbolic thermoelastic system with dissipative boundary, *J. Thermal Stresses* 30 (2007) 1–14.
- [21] J.E. Munoz Rivera, D. Andrade, Exponential decay of non-linear wave equation with a viscoelastic boundary condition, *Math. Methods Appl. Sci.* 23 (2000) 41–61.
- [22] S. Murakami, Exponential asymptotic stability of scalar linear Volterra equations, *Differential Integral Equations* 4 (1991) 519–525.
- [23] L. Nappa, Spatial decay estimates for the evolution equations of linear thermoelasticity without energy dissipation, *J. Thermal Stresses* 21 (1998) 581–592.
- [24] R. Nibbi, S. Polidoro, Exponential decay for Maxwell equations with a boundary memory condition, *J. Math. Anal. Appl.* 302 (2005) 30–55.
- [25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, *Appl. Math. Sci.*, vol. 44, Springer-Verlag, New York, 1983.
- [26] G. Propst, J. Prüss, On wave equations with boundary dissipation of memory type, *J. Integral Equations Appl.* 8 (1996) 99–123.
- [27] R. Quintanilla, Existence in thermoelasticity without energy dissipation, *J. Thermal Stresses* 25 (2002) 195–202.
- [28] R. Quintanilla, Some remarks on growth and uniqueness in thermoelasticity, *Int. J. Math. Sci.* 2003 (2003) 617–623.
- [29] R. Quintanilla, R. Racke, Stability in thermoelasticity of type III, *Discrete Contin. Dyn. Syst. Ser. B* 3 (2003) 383–400.
- [30] R. Quintanilla, B. Straughan, Growth and uniqueness in thermoelasticity, *Proc. Roy. Soc. London A* 456 (2000) 1419–1429.
- [31] R. Quintanilla, B. Straughan, Energy bounds for some non-standard problems in thermoelasticity, *Proc. Roy. Soc. London A* 461 (2005) 1147–1162.
- [32] M. Reissig, Y.-G. Wang, Cauchy problems for linear thermoelastic systems of type III in one space variable, *Math. Methods Appl. Sci.* 28 (2005) 1359–1381.
- [33] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, *Appl. Math. Sci.*, vol. 68, Springer-Verlag, New York, 1988.
- [34] L. Yang, Y.-G. Wang, Well-posedness and decay estimates for Cauchy problems of linear thermoelastic systems of type III in 3-D, *Indiana Univ. Math. J.* 55 (2006) 1333–1362.
- [35] X. Zhang, E. Zuazua, Decay of solutions of the system of thermoelasticity of type III, *Commun. Contemp. Math.* 5 (2003) 25–84.